THE MILNOR–CHOW HOMOMORPHISM REVISITED

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Abstract. The aim of this note is to give a simplified proof of the surjectivity of the natural Milnor–Chow homomorphism $\rho : K_n^M(A) \to CH^n(A,n)$ between Milnor K–theory and higher Chow groups for essentially smooth (semi–)local k–algebras A with k infinite. It implies the exactness of the Gersten resolution for Milnor K–theory at the generic point. Our method uses the Bloch-Levine moving technique and some properties of the Milnor K–theory norm for fields.

Introduction

In [4, 3] the surjectivity of the natural Milnor–Chow homomorphism

$$\rho : K_n^M(A) \to CH^n(A,n)$$

between Milnor K–theory and higher Chow groups for any essentially smooth (semi–)local k–algebra A with k infinite was shown. This morphism associates to a symbol $\{f_1, \dots, f_n\}$ the graph cycle of the map $f = (f_1, \dots, f_n)$.
In this note we want to give a very simple argument which uses two basic ingredients. The first is a new argument derived from fairly elementary properties of the norm–map for the Milnor K–theory of rings which were sketched in [7] and build up on the theory of Bass and Tate [1] (see section 2). The idea in [7] was to use a Milnor K–group which is not induced directly from a ring (or algebra) but only from certain generic elements of a ring. The same technique can also be used to show the Gersten conjecture for Milnor K–theory of regular (semi–)local rings [7]. The second input is a standard application of the easy moving lemma of Bloch–Levine [2, 8] which implies that we can restrict to the case of cycles with smooth components. This was also used in the proof in [3].
Our main theorem is:

Theorem 0.1. *Let A be an essentially smooth (semi–)local k–algebra with infinite residue fields. Then the homomorphism $\rho : K_n^M(A) \to CH^n(A,n)$ is surjective for $n \geq 1$.*

Here for a field k we say that a k–algebra A is essentially smooth if A is the localization of a smooth affine k–algebra. In fact under the conditions of the theorem one can show ρ is bijective [7]. This theorem has a few beautiful applications:

Corollary 0.2. *Let A be as above and $X = Spec(A)$ integral (i.e., A a domain with quotient field F). Then the Gersten resolution for Milnor K–theory is exact*

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at the generic point:

$$K_n^M(A) \xrightarrow{i_*} K_n^M(F) \xrightarrow{T} \bigoplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \to \dots,$$

i.e., $\ker(T) = \mathrm{im}(i_*)$*, where* T *is the tame symbol.*

The exactness of this complex is well known in codimensions $p \geq 1$ by the work of Gabber and Rost [10] and follows in degree zero with the same proof as in [3] by comparing with the corresponding sequence for higher Chow groups [2]. Note that the work of Kerz [7] implies also the Gersten conjecture, i.e. the injectivity of i_* for such k–algebras A. There is another nice application to étale cohomology:

Corollary 0.3. *Assume the Bloch–Kato conjecture* [13]*. Let* A *be a (semi–)local ring containing an infinite field and* $l > 0$ *prime to* $char(A)$*. Then the graded ring* $H^*_{\mathrm{et}}(A, \mu_l^{\otimes *})$ *is generated by elements of degree one.*

Proof. First assume that A is essentially smooth over an infinite field. The Bloch–Kato conjecture implies that we have an isomorphism

$$CH^n(A,n)/l \xrightarrow{\cong} H^n_{\mathrm{et}}(A, \mu_l^{\otimes n})$$

for any l prime to $char(A)$. Composing with ρ we get a surjective ring homomorphism $K_n^N(A)/l \to H^n_{\mathrm{et}}(A, \mu_l^{\otimes n})$ which shows the corollary in this case, because Milnor K–theory is generated in degree one.
Let A be arbitrary. By a direct limit argument we can assume A to be a localization of an affine algebra. Now Hoobler's trick [5] can be applied: There is a Henselian pair (A', I) with A' essentially smooth and $A = A'/I$. In this situation $H^n_{\mathrm{et}}(A, \mu_l^{\otimes n})$ and $H^n_{\mathrm{et}}(A', \mu_l^{\otimes n})$ are isomorphic [5]. The commutative diagram

$$\begin{array}{ccc} K_n^M(A')/l & \xrightarrow{nat} & K_n^M(A)/l \\ \rho\big\downarrow & & \rho\big\downarrow \\ H^n_{\mathrm{et}}(A', \mu_l^{\otimes n}) & \xrightarrow[nat]{} & H^n_{\mathrm{et}}(A, \mu_l^{\otimes n}) \end{array}$$

implies immediately that $\rho : K_n^N(A)/l \to H^n_{\mathrm{et}}(A, \mu_l^{\otimes n})$ is surjective. □

Corollary 0.4 (Bloch)**.** *Again assuming the Bloch–Kato conjecture, let* $X/\mathbb{C}$ *be a variety and* $\xi \in H^i(X, \mathbb{Z})$ *an element of prime exponent* l*. Fix some points* $x_1, \dots, x_n \in X$*. Then there exists an effective divisor* $D \subset X$ *such that* ξ *restricted to* $X - D$ *vanishes and* $x_j \notin D$ *for all* $j = 1, \dots, n$*.*

Proof. This is essentially the same argument as in the proof of Corollary 7.7 of [12]. □

1. The Milnor–Chow map ρ

1.1. **Higher Chow groups.** S. Bloch [2] defined *higher Chow groups* as a candidate for motivic cohomology, i.e. an algebraic singular (co)homology. They form

a Borel–Moore homology theory for schemes over a field k, which we fix from now on. In order to define them we use the algebraic n–cube

$$\square^n = (\mathbb{P}^1_k \setminus \{1\})^n.$$

The n–cube has 2^n codimension one faces, defined by $x_i = 0$ and $x_i = \infty$ for $1 \leq i \leq n$. An integral subvariety $W \subseteq \square^n$ of codimension p is called admissible if its intersection with all faces is again of codimension p or empty. For each face $F = \{x_i = 0\}$ or $F = \{x_i = \infty\}$ we have a pull–back map ∂_i^0 resp. ∂_i^∞ which sends a subvariety $W \subseteq \square^n$ to the intersection product of cycles $W \cdot F$ with appropriate multiplicities in the sense of Serre's Tor–formula. A total differential is given by

$$\partial = \sum_{i=1}^{n} (-1)^{i-1} (\partial_i^0 - \partial_i^\infty).$$

Let X be a quasi–projective variety over k (standard techniques allow to extend this definition to equidimensional schemes over k and even, but much harder, to schemes over Dedekind rings). The notion of faces, restriction maps, and differentials extends to $\square^n_X = X \times_k \square^n$. $Z^p_c(X, n)$ is defined to be the quotient of the group of admissible cycles of codimension p in $X \times \square^n$ by the group of degenerate cycles as defined in [11], p.180 (where they are denoted by $d^p(X, n)$). Let $CH^p(X, n)$ be the n–th homology of the complex $Z^p_c(X, \cdot)$ with differential ∂.

1.2. **Milnor K–theory.** Milnor K–theory of a ring A is defined as the quotient

$$T(A)/S(A)$$

of the free graded tensor algebra $T(A) = \mathbb{Z} \oplus A^\times \oplus A^\times \otimes A^\times \oplus \cdots$ over the units $A^\times$ of A by the ideal $S(A)$ generated by the degree two relations of the form $(f, 1-f)$ for all f with $f, 1-f \in A^\times$ and $(f, -f)$ for all $f \in A^\times$. Note that in the case of fields or (semi–)local rings with large residue fields the relation $(f, -f)$ follows from the usual Steinberg relation $(f, 1-f)$.

1.3. **The map $K^M_n(A) \to CH^n(A, n)$.** Now we consider the special case where A is a localization of an affine k–algebra with k an arbitrary ground field. Denote by $CH^p(A, n)$ the higher Chow groups of $\mathrm{Spec}(A)$. In particular we have the series of abelian groups $CH^n(A, n)$. To any tuple $f = (f_1, \ldots, f_n)$ of elements $f_i \in A^\times$ we can associate a map

$$f = (f_1, \ldots, f_n) : \mathrm{Spec}(A) \to (\mathbb{P}^1)^n$$

and hence by restricting to the cube a graph cycle

$$\Gamma_f = \mathrm{graph}(f_1, \ldots, f_n) \cap \square^n_A.$$

Since such graph cycles have no boundary, we immediately get a map

$$\rho : (A^\times)^n \to CH^n(A, n).$$

One can show that ρ preserves bilinearity, is skew–commutative, and obeys the Steinberg relations $\rho(f, 1-f, f_3, \ldots, f_n) = 0$ and $\rho(f, -f, f_3, \ldots, f_n) = 0$ [3]. Therefore it descends to a well–defined homomorphism

$$\rho : K^M_n(A) \to CH^n(A, n)$$

for all $n \geq 0$. If A is essentially smooth $CH^*(A,*)$ has a ring structure and ρ becomes a ring homomorphism. In the special case where A is a field F the following result is classical.

Theorem 1.1 (Nesterenko/Suslin, Totaro). *ρ is an isomorphism for every field F.*

Proof. Totaro's proof [11] uses cubical higher Chow groups as defined above. He shows that any cycle $Z \in CH^n(F,n)$ is equivalent (cobordant) to a norm–cycle which has all coordinate entries in F. This already gives the surjectivity of ρ. The inverse map ρ^{-1} is defined using the norm as follows: By linearity it is sufficient to define ρ^{-1} for Z irreducible. In this case we choose a minimal finite field extension L/F such that Z corresponds to an L–valued point $(z_1,\dots,z_n)$. Then $\rho^{-1}(Z) = N_{L/F}(\{z_1,\dots,z_n\})$ as an element of $K_n^M(F)$, where $N_{L/F}$ is the norm map of Bass and Tate [1]. $\square$

2. Symbols in general position

The main result of this section is Proposition 2.8 which in some sense represents the idea that for good extensions of (semi–)local rings there should be norms of Milnor K–groups as in the field case. In fact such norms can be constructed by an extension of the methods described below [7].

2.1. **The group $K_n^t(A)$.** Let A be a (semi–)local UFD and $F = Q(A)$ its quotient field. The group $K_n^t(A)$, we are going to define, should be thought of as the proper Milnor K–group of the ring $A[t]_S$, where S denotes the multiplicative system of all monic polynomials.

Definition 2.1. *An n–tuple of rational functions*

$$\left(\frac{p_1}{q_1},\frac{p_2}{q_2},\dots,\frac{p_n}{q_n}\right) \in F(t)^n$$

with $p_i, q_i \in A[t]$ for $i = 1,\dots n$ is called feasible if

1. *The highest nonvanishing coefficients of p_i, q_i are invertible in A for $i = 1,\dots,n$.*
2. *For every irreducible factor u of p_i or q_i and v of p_j or q_j ($i,j = 1,\dots,n$, $i \neq j$) u is either equivalent or coprime to v.*

Before stating the definition of $K_n^t(A)$ we have to replace ordinary tensor product.

Definition 2.2. *Define*

$$\mathcal{T}_n^t(A) = \mathbb{Z} < \{(p_1,\dots,p_n) | (p_1,\dots,p_n) \text{ feasible, } p_i \in A[t] \text{ irreducible or unit}\} > /L$$

Here L denotes the subgroup generated by elements

$$(p_1,\dots,ap_i,\dots,p_n) - (p_1,\dots,a,\dots,p_n) - (p_1,\dots,p_i,\dots,p_n)$$

with $a \in A^\times$.

By bilinear factorization the element

$$(p_1, \ldots, p_n) \in \mathcal{T}_n^t(A)$$

is defined for every feasible n–tuple with $p_i \in F(t)$.
Now define the subgroup $St \subset \mathcal{T}_n^t(A)$ to be generated by feasible n–tuples

$$(p_1, \ldots, p, 1-p, \ldots, p_n) \tag{1}$$

and

$$(p_1, \ldots, p, -p, \ldots, p_n) \tag{2}$$

with $p_i, p \in F(t)$.

Definition 2.3. *Define*

$$K_n^t(A) = \mathcal{T}_n^t(A)/St$$

We denote the image of $(p_1, \ldots, p_n)$ in $K_n^t(A)$ by $\{p_1, \ldots, p_n\}$.

2.2. **The tame symbol.** Recall that Milnor constructed so called tame symbols

$$\partial_\pi : K_n^M(F(t)) \longrightarrow K_{n-1}^M(F[t]/(\pi))$$

for every irreducible $\pi \in F[t]$ [9] – in fact this construction works for all discrete valuation rings in contrast to our generalization below.

Proposition 2.4 (Tame symbol)**.** *For every irreducible, monic polynomial $\pi \in A[t]$ and $n > 0$ one has a unique well defined tame symbol*

$$\partial_\pi : K_n^t(A) \longrightarrow K_{n-1}^M(A[t]/(\pi))$$

which satisfies

$$\partial_\pi : \{\pi, x_2, \ldots, x_n\} \mapsto \{\bar{x}_2, \ldots, \bar{x}_n\} \tag{3}$$

for $x_i \in A[t]$ and x_i coprime to π.
For $\pi = 1/t$ there is a similar tame symbol

$$\partial_\pi : K_n^t(A) \longrightarrow K_{n-1}^M(A)$$

which satisfies (3) for $x_i \in A[1/t]$.

Proof. Assume $\pi \in A[t]$. Uniqueness is easy to check. In order to show existence, introduce according to an idea of Serre a formal skew–commutative element ξ with $\xi^2 = \xi\{-1\}$ and $\deg(\xi) = 1$. Define a formal map (which is clearly not well defined)

$$\theta_\pi : \mathcal{T}_*^t(A) \longrightarrow K_*^M(A[t]/(\pi))[\xi]$$

by

$$\theta_\pi(u_1\pi^{i_1}, \ldots, u_n\pi^{i_n}) = (i_1\xi + \{\bar{u}_1\}) \cdots (i_n\xi + \{\bar{u}_n\}) \; .$$

We define ∂_π by taking the (right–)coefficient of ξ. This is a well defined homomorphism. So what remains to be shown is that ∂_π factors over the Steinberg relations.
Let $x = (\pi^i u, -\pi^i u)$ be feasible, then

$$\begin{aligned} \theta_\pi(x) &= (i\xi + \{\bar{u}\})(i\xi + \{-\bar{u}\}) \\ &= i\xi\{-1\} - i\xi\{\bar{u}\} + i\xi\{-\bar{u}\} + \{\bar{u}, -\bar{u}\} = 0 \; . \end{aligned}$$

For $i > 0$ and $x = (\pi^i u, 1 - \pi^i u)$ feasible one has

$$\theta_\pi(x) = (i\xi + \{\bar{u}\})\{1\} = 0 \ .$$

For $i < 0$ and $x = (\pi^i u, 1 - \pi^i u)$ feasible one has

$$\begin{aligned} \theta_\pi(x) &= (i\xi + \{\bar{u}\})(i\xi + \{-\bar{u}\}) \\ &= i\xi\{-1\} + i\xi\{-\bar{u}\} - i\xi\{\bar{u}\} + \{\bar{u}, -\bar{u}\} = 0 \ . \end{aligned}$$

□

The tame symbols from Proposition 2.4 are compatible with the corresponding symbols of the quotient field of A. This is the content of the next lemma.

Lemma 2.5. *Fix either an irreducible, monic $\pi \in A[t]$ as above and let $B = A[t]/(\pi)$ and $L = Q(B)$ or set $\pi = 1/t$, $B = A$ and $L = F$. The square*

$$\begin{array}{ccc} K_n^t(A) & \xrightarrow{\partial_\pi} & K_{n-1}^M(B) \\ \downarrow & & \downarrow \\ K_n^M(F(t)) & \xrightarrow[\partial_\pi]{} & K_{n-1}^M(L) \end{array}$$

is commutative.
Moreover for monic $\pi \in F[t]$ but $\pi \notin A[t]$ the composition

$$K_n^t(A) \longrightarrow K_n^M(F(t)) \xrightarrow{\partial_\pi} K_{n-1}^M(F[t]/(\pi))$$

vanishes.

Proposition 2.6. *If the residue fields of A are infinite the map*

$$\oplus_\pi \partial_\pi : K_n^t(A) \longrightarrow \oplus_\pi K_{n-1}^M(A[t]/(\pi))$$

is surjective, where the sum is over all monic, irreducible $\pi \in A[t]$.

In fact the kernel of $\oplus_\pi \partial_\pi$ is precisely $K_n^M(A)$, but this is more difficult to show [7].
Proof. Consider the filtration $L_d \subset K_n^t(A)$, where L_d is generated by the feasible $(x_1, \dots, x_n)$ with $x_i \in A[t]$ of degree at most d. One has to show

$$\oplus_{\deg(\pi)=d} \partial_\pi : L_d \longrightarrow \oplus_{\deg(\pi)=d} K_{n-1}^M(A[t]/(\pi))$$

is surjective. Fix π of degree d. For a symbol $\xi = \{\bar{x}_2, \dots, \bar{x}_n\} \in K_{n-1}^M(A[t]/(\pi))$ we can according to the following sublemma suppose without restriction that $\zeta = \{\pi, x_2, \dots, x_n\} \in K_n^t(A)$ is well defined assuming x_i to be choosen of degree $d-1$. As we have $\partial_{\pi'}(\zeta) = 0$ for $\pi' \neq \pi$, $\deg(\pi') = d$, and $\partial_\pi(\zeta) = \xi$, this proves the proposition.

Sublemma 2.7 (Gabber)**.** *Given monic $y_1, \dots, y_k \in A[t]$ and an arbitrary $x \in A[t]$ coprime to π there exists a factorization*

$$x \equiv x' x'' \ mod \ (\pi)$$

such that $x', x'' \in A[t]$ have invertible highest coefficients, $\deg(x') = \deg(x'') = d-1$ *and x', x'' are coprime to y_j for $j = 1, \dots, k$.*

Proof. Using the Chinese remainder theorem and reduction modulo all maximal ideals we can assume that A is an infinite field. The moduli space of factorizations $x \equiv x'x''$ mod (π) is a nonempty Zariski open subset of $\mathbb{A}^d_A$. As finite intersections of such subsets contain a rational point, the sublemma is proven. $\square$

2.3. **Norms.** With the notation as above ($B = A[t]/(\pi)$, $F = Q(A)$, $L = Q(B)$) let $i : A \to F$ and $j : B \to L$ be the natural embeddings. For the convenience of the reader we recall the construction of norms

$$N_{L/F} : K_n^M(L) \to K_n^M(F)$$

from [1].
Given $\xi \in K_n^M(L)$ choose $\zeta \in K_{n+1}^M(F(t))$ such that $\partial_{\pi'}(\zeta) = 0$ for $\pi' \neq \pi$ and $\partial_\pi(\zeta) = \xi$. Set $N_{L/F}(\xi) = -\partial_{1/t}(\zeta)$. Kato showed this norm depends only on the isomorphism class of (L, ξ) over F and is functorial [6].

Proposition 2.8. *We have*

$$N_{L/F}(\mathrm{im}(j_*)) \subset \mathrm{im}(i_*)$$

with i_*, j_* the homomorphisms induced on Milnor K–groups.

Proof. Given $\xi \in K_n^M(B)$ choose by Lemma 2.6 $\zeta \in K_{n+1}^t(A)$ such that $\partial_{\pi'}(\zeta) = 0$ for $\pi \neq \pi' \in A[t]$ and $\partial_\pi(\zeta) = \xi$. Set $\xi' = -\partial_{1/t}(\zeta) \in K_n^M(A)$. It follows from Lemma 2.5 that

$$N_{L/F}(j_*(\xi)) = i_*(\xi') \ .$$

$\square$

3. Proof of Theorem 0.1

Assume that $[Z] \in CH^n(A, n)$ is a higher Chow cycle. We want to construct an element $\xi \in K_n^M(A)$ such that $\rho(\xi) = [Z]$.

Lemma 3.1. *Z is cobordant to a sum of irreducible cycles Z' such that*

(1) *$Z' \subset \square^n_A$ does not intersect any face.*
(2) *With the coordinate functions $t_1, \ldots, t_n \in \mathcal{O}_{Z'}$ one has $A[t_1, \ldots, t_i]$ essentially smooth over k and finite over A for every $1 \le i \le n$.*
(3) *$\mathcal{O}_{Z'} = A[t_1, \ldots, t_n]$.*

Proof. This follows immediately from the "easy moving lemma" of Bloch and Levine [8, chap. II, 3.5] and is also applied and explained in [3]. $\square$.
Without loss of generality we may therefore assume that Z is irreducible and already in good position as in the lemma. Look at the following diagram:

$$\begin{array}{ccc} K_n^M(A) & \xrightarrow{i_*} & K_n^M(F) \\ \rho\big\downarrow & & \rho\big\downarrow \\ CH^n(A,n) & \xrightarrow[i_*]{} & CH^n(F,n) \end{array}$$

where – by abuse of notation – we use the same symbols ρ and i_* for the corresponding maps of rings or fields and F is the quotient field of A. Since ρ is an isomorphism on the level of fields, we know that there is an element $\tau \in K_n^M(F)$ such that $\rho(\tau) = i_*[Z]$. By the description of ρ^{-1} in Totaro's proof of Theorem 1.1, we know that one has $\tau = N_{L/F}(\{t_1, \dots, t_n\})$ where L is the quotient field of $\mathcal{O}_Z$ and $N_{L/F}$ is the norm on Milnor K–theory of fields. Now look at the consecutive extensions

$$A \subset A[t_1] \subset A[t_1, t_2] \subset \dots \subset A[t_1, \dots, t_i] \subset \dots$$

These rings are all essentially smooth and hence factorial. Each extension is of the type

$$A[t_1, \dots, t_{i+1}] = A[t_1, \dots, t_i][t]/(\pi_{i+1}).$$

Therefore we may apply Proposition 2.8 and conclude that there is an element ξ with $i_*(\xi) = \tau$. But the map $i_* : CH^n(A, n) \to CH^n(F, n)$ is injective by [2] and therefore we have $\rho(\xi) = [Z]$, since $i_*(\rho(\xi)) = i_*[Z]$. $\square$.

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